Announcements

1) Practice problems for HW #4 available later today

Recall: If A is in $M_n(\mathbb{R})$, then A is invertible precisely when $\det(A) \neq 0$.

Calling sequence in Wolfram Alpha

det(A)

Q: How do you find A^{-1} if you know A is invertible?

A: Consider the $n \times 2n$ matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}.$$

If A is invertible, put this matrix in row-reduced echelon form.

You will get

$$\begin{bmatrix} I_n & A^{-1} \end{bmatrix}.$$

Observation: A is invertible precisely when $\mathrm{rref}(A) = I_n$

Example 1: (2×2)

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

and $\det(A) = ad - bc \neq 0$,

then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So if

$$A = \begin{bmatrix} -1 & 13 \\ 2 & -56 \end{bmatrix},$$

then $\det(A) = 30 \neq 0$, so

$$A^{-1} = \frac{1}{30}\begin{bmatrix} -56 & -13 \\ -2 & -1 \end{bmatrix}$$

Example 2: (4×4)

Let $A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 15 & 3 & 10 \\ 8 & -2 & 6 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

By computer or tedious calculation,

$\det(A) = 16$, so

A is invertible.

We find A^{-1} by row-reducing $\begin{bmatrix} A & I_4 \end{bmatrix}$. We get

$$\left[\begin{array}{c|cccc} I_4 & 4 & \frac{25}{2} & \frac{69}{4} & -\frac{141}{4} \\ & 1 & 7/2 & 19/4 & -39/4 \\ & -5 & -31/2 & -85/4 & 175/4 \\ & 0 & -1/2 & -3/4 & 3/2 \end{array}\right]$$

$= A^{-1}$

Solving Linear Equations via Inverses

If A is in $M_n(\mathbb{R})$, x, b are in $\mathbb{R}^n$, solve

$$Ax = b.$$

If A is invertible, multiply both sides by A^{-1} to get

$$\underbrace{A^{-1}A}_{= I_n} x = A^{-1}b$$

$$= I_n x = A^{-1}b$$

$$= x = A^{-1}b$$

Observe! If x is an n-vector,

$$I_n x = x.$$

Example 3:

Solve

$$3x_1 - 2x_2 + 4x_3 = 0$$
$$-8x_1 + 10x_2 - 15x_3 = 1$$
$$-5x_1 + 8x_2 + 13x_3 = 2$$

Same as

$$\underbrace{\begin{bmatrix} 3 & -2 & 4 \\ -8 & 10 & -15 \\ -5 & 8 & 13 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{b}$$

$\det(A) = 336 \neq 0$,

so A is invertible.

Find A^{-1}:

$$\operatorname{rref}\begin{bmatrix} A & I_3 \end{bmatrix}.$$

We get

$$\left[\begin{array}{c|ccc} I_3 & \frac{125}{168} & \frac{29}{168} & -\frac{5}{168} \\ & \frac{176}{336} & \frac{59}{336} & \frac{13}{336} \\ & -\frac{1}{24} & -\frac{1}{24} & \frac{1}{24} \end{array}\right]$$

$= A^{-1}$

In order to find x,

we calculate

$$A^{-1}b = x$$

$$\begin{bmatrix} \frac{125}{168} & \frac{29}{168} & -\frac{5}{168} \\ \frac{176}{336} & \frac{59}{336} & \frac{13}{336} \\ -1/24 & -1/24 & 1/24 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} \frac{125}{168} & \frac{29}{168} & -\frac{5}{168} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{29}{168} - \frac{10}{168} = \frac{19}{168}$$

$$X_2 = \begin{bmatrix} \frac{176}{336} & \frac{59}{336} & \frac{13}{336} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{59}{336} + \frac{26}{336}$$

$$= \frac{85}{336}$$

$$X_3 = \begin{bmatrix} -1/24 & -1/24 & 1/24 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{24} + \frac{2}{24}$$

$$= \frac{1}{24}$$

Digression: Transpose of a product.

$$\boxed{(AB)^t = B^t A^t}$$

for A an $m\times n$ matrix

and B an $n\times k$ matrix.

Properties of Inverses

Let A and B be invertible in $M_n(\mathbb{R})$

1) AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

2) $(A^t)^{-1} = (A^{-1})^t$

Why are these facts true?

1) $AB(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$ (assoc.)

$= A(I_n)A^{-1}$

$= AA^{-1}$

$= I_n$

So then $(AB)^{-1} = B^{-1}A^{-1}$!

2) $$\underbrace{A^t (A^{-1})}_{B}{}^t$$

$$= (A^{-1} A)^t$$ (properties of transpose)

$$= (I_n)^t$$

$$= I_n$$

This shows $(A^t)^{-1} = (A^{-1})^t$

Example 4:

Find the inverse of

$$A^2 B^t (C^{-1})^3.$$

provided A, B, C are invertible $n \times n$ matrices.

$$\left(A^2 B^t (C^{-1})^3\right)^{-1}$$

$$= \left((C^{-1})^3\right)^{-1} \cdot (B^t)^{-1} (A^2)^{-1}$$

inverse reverses the product

$$= \left(\underbrace{(C^3)^{-1}}_{\text{property 1}}\right)^{-1} \underbrace{(B^{-1})^t}_{\text{property 2}} \underbrace{(A^{-1})^2}_{\text{property 1}}$$

$$= \boxed{C^3 (B^{-1})^t (A^{-1})^2}$$

Properties of Determinants

A, B in $M_n(\mathbb{R})$.

1) $\det(AB) = \det(A)\det(B)$

2) If A is invertible,

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

3) $\det(A^t) = \det(A)$

4) If B is obtained from A by multiplying a row or column of A by a constant c, then

$$\boxed{\det(B) = c\det(A)}$$

5) If B is obtained from A by interchanging two rows or columns, then $\boxed{\det(B) = -\det(A)}$